Fluctuation-induced transport in a spatially symmetric periodic potential

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We present an analytical investigation of the fluctuation-induced transport of Brownian particles in a deterministic spatial symmetrical periodic potential subject to Gaussian noises. We found that directed motion of the Brownian particles can be induced by the correlation between a multiplicative white noise and an additive white noise. The direction of current is determined by the sign of the noise correlation.

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Since the publication of Magnasco's 1993 work [1] on forced thermal ratchets, many researchers invested time on thermal ratchets (TR) [2–5]. To biologists, TR has been investigated as a possible prototype of a molecular motor [2]. To physicists, TR's importance relies on its relevance to fluctuation-induced transport [3–5]. From Refs. [1–4], one can conclude that there exists no fluctuation-induced transport in a spatial symmetrical periodic potential if all odd moments of the noise are zero. It should noted that the studies in [1–5] are subject to the following restrictions: (1) No quantum effect is included; (2) only the overdamped Brownian motion is considered with inertia effect neglected; (3) only uncorrelated noises are studied. In 1997, Ref. [6] considered the quantum effect. And, in 1998, Ref. [7] gave a study of underdamped ratchets. In recent years, much progress has been obtained of correlated noises in other systems. For example, correlated noises in bistable systems can induce giant suppression of the activation rate [8], re-entrant phase transition [9,10], and the coexistence of the suppression and resonance of activation [11]. In a periodic potential system, we expect, correlation between noises should produce similar significant impact. This Brief Report presents a study that shows that fluctuation-induced transport exists as a noise correlation effect when all the odd moments of noises are zero and the periodic potential is symmetrical in space. This effect should be experimentally observable in transport in Josephson junctions. In the power spectrum of quasiparticles in Josephson junction, we discovered a pair of negatively correlated noises with correlation coefficient $\lambda=-\sqrt{6}/3$ [12]. This correlation has significant implication on the I-V characteristics [12]. Other applications of this correlation will be presented in a future publication. In the remainder of this Brief Report, we will first present general solution for the transport in periodic potential subject to a multiplicative noise and an additive noise. Then we calculate the fluctuation-induced current caused by the correlated multiplicative and additive noises. At the end, we discuss the experimental implications of this theoretical work.

Consider an overdamped Brownian particle in a periodic potential $U(x)$ that possesses spatial symmetry, $U(x+L)=U(x)$ with L being the spatial period. The stochastic dynamics is governed by the Langevin equation

$$\dot{x}=-U'(x)+g_1(x)\xi(t)+g_2(x)\eta(t). \tag{1}$$

In Eq. (1), $\xi(t)$ and $\eta(t)$ are two Gaussian white noises with zero mean. They are correlated in the following manner:

$$\langle\xi(t)\xi(t')\rangle=2Q\delta(t-t'),$$
$$\langle\eta(t)\eta(t')\rangle=2D\delta(t-t'), \tag{2a}$$

$$\langle\xi(t)\eta(t')\rangle=\langle\eta(t)\xi(t')\rangle=2\lambda\sqrt{QD}\,\delta(t-t'). \tag{2b}$$

Here, λ denotes the intensity of correlation between $\xi(t)$ and $\eta(t)$. Q and D are the noise intensities. $g_1(x)$ and $g_2(x)$, multiplicative functions, can be nonlinear in general. We assume that Eq. (1) is a Stratonovich stochastic differential equation. Employing the technique we developed earlier in Refs. [13], [14], Eq. (1) with Eq. (2) can be transformed into an equivalent form

$$\dot{x}=-U'(x)+G(x)\Gamma(t) \tag{3}$$

that has the same Fokker-Planck equation as Eq. (1). In Eq. (3), $\Gamma(t)$ is Gaussian white noise with zero mean and the following correlation:

$$\langle\Gamma(t)\Gamma(t')\rangle=2\delta(t-t'), \tag{4}$$

and $G(x)$ is determined as

$$G(x)=\{Qg_1(x)^2+2\lambda\sqrt{QD}g_1(x)g_2(x)+Dg_2(x)^2\}^{1/2}. \tag{5}$$

The Fokker-Planck equation corresponding to Eq. (3) with Eq. (4) can be written as [15]

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial}{\partial x}A(x)P(x,t)+\frac{\partial^2}{\partial x^2}B(x)P(x,t), \tag{6}$$

where

$$A(x)=-U'(x)+G(x)G'(x) \tag{7}$$

and

$$B(x)=G(x)^2. \tag{8}$$

By virtue of the generalized potential $\Phi(x)$ defined as

$$\Phi(x)=-\int_0^x \frac{A(y)}{B(y)}dy \tag{9}$$

The Fokker-Planck operator can be written in the form

$$L_{FP}=-\frac{\partial}{\partial x}A(x)+\frac{\partial^2}{\partial x^2}B(x)=\frac{\partial}{\partial x}e^{-\Phi(x)}\frac{\partial}{\partial x}e^{\Phi(x)}B(x). \tag{10}$$

The stationary current J determined by Fokker-Planck equation (6) reads

$$J=-e^{-\Phi(x)}\frac{\partial}{\partial x}e^{\Phi(x)}B(x)P_{st}(x). \tag{11}$$

Integrating Eq. (11) from x to $x+L$, we have

$$J\int_x^{x+L}e^{\Phi(y)}dy=-[e^{\Phi(x+L)}B(x+L)P_{st}(x+L)-e^{\Phi(x)}B(x)P_{st}(x)]. \tag{12}$$

Assuming that $g_1(x)$ *and* $g_2(x)$ are also periodic in x with period L, then $B(x+L)=B(x)$. Further, the periodic boundary condition $P(x+L)=P(x)$ leads to the following stationary solution:

$$P_{st}(x)=JB(x)^{-1}e^{-\Phi(x)}\int_x^{x+L}e^{\Phi(y)}dy[1-e^{\Phi(x+L)-\Phi(x)}]^{-1}. \tag{13}$$

Normalizing the probability $p_{st}(x)$ within a period, e.g., $[0,L]$, we get from Eq. (13)

$$1=J[1-e^{\Phi(x+L)-\Phi(x)}]^{-1}\times\int_0^L B(x)^{-1}e^{-\Phi(x)}dx\int_x^{x+L}e^{\Phi(y)}dy.$$

This leads to the general current formula

$$J=[1-e^{\Phi(x+L)-\Phi(x)}]\times\left[\int_0^L B(x)^{-1}e^{-\Phi(x)}dx\int_x^{x+L}e^{\Phi(y)}dy\right]^{-1}, \tag{14}$$

which is one of the major results in this Brief Report

Now we use Eq. (14) to evaluate the fluctuation-induced current in a model system to illustrate the new physics. The periodic potential $U(x)$ is given by

$$U(x)=\begin{cases}2dx/L-2(n-1)d & \text{when } (n-1)L\leqslant x\leqslant(2n-1)L/2\\ -2dx/L+2nd & \text{when } (2n-1)L/2\leqslant x\leqslant nL.\end{cases}$$

The two multiplicative functions $g_1(x)$ and $g_2(x)$ are

$$g_1(x)=\begin{cases}C & \text{when } (n-1)L\leqslant x\leqslant(2n-1)L/2\\ -C & \text{when } (2n-1)L/2\leqslant x\leqslant nL\end{cases}$$

and

$$g_2(x)=1.$$

In this, $\xi(t)$ is a multiplicative noise while $\eta(t)$ is an additive noise. Carrying out the integrations in Eq. (14), a closed analytical form of the current can be obtained:

$$J=\frac{1-\exp[-8\lambda\sqrt{QD}Cd/(k_1k_2)]}{A_1+A_2+A_3+A_4+A_5+A_6}, \tag{15}$$

where

$$k_1=QC^2+2\lambda\sqrt{QD}C+D,$$

$$k_2=QC^2-2\lambda\sqrt{QD}C+D,$$

$$A_1=\frac{L^2}{4d}\left[\frac{k_1}{d}e^{d/k_1}-\frac{k_1}{d}-1\right],$$

$$A_2=\frac{L^2}{4d^2}(k_1k_2)^{1/2}e^{-d/k_2}(e^{-d/k_1}-1)(e^{-d/k_2}-1),$$

$$A_3=\frac{L^2}{4d}\left[e^{2d/k_1}+\frac{k_1}{d}e^{d/k_1}(1-e^{d/k_1})\right],$$

$$A_4=\frac{L^2}{4d}\left[\frac{k_2}{d}(e^{-d/k_2}-1)+1\right]$$

$$A_5=\frac{L^2}{4d^2}(k_1k_2)^{1/2}e^{2d/k_1}e^{d/k_2}(e^{d/k_1}-1)(e^{d/k_2}-1),$$

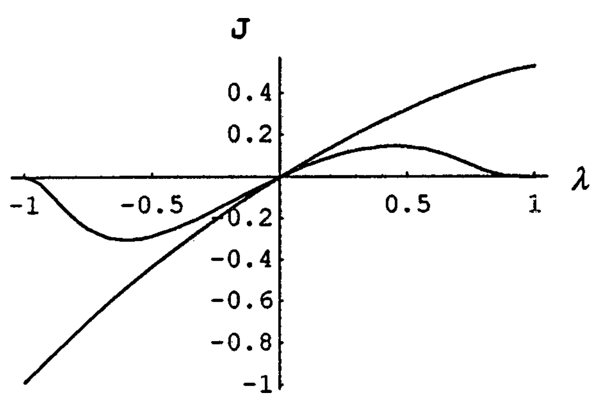

FIG. 1. Steady current J vs the noise correlation strength λ. The curve that possess extrema for $Q=10$; the other for $Q=40$; $L=1$, $C=\frac{1}{4}$, $d=\frac{1}{2}$, $D=0.3$.

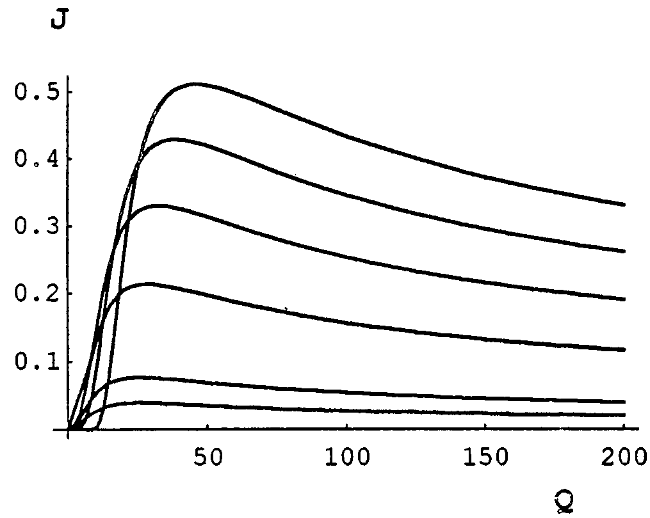

FIG. 2. Steady current J vs the multiplicative noise strength Q. From top to bottom, the curves corresponding to the value of noise correlation strength λ:0.9, 0.7, 0.5, 0.3, 0.1, 0.05, respectively; $L=1$, $C=\frac{1}{4}$, $d=\frac{1}{2}$, $D=0.3$.

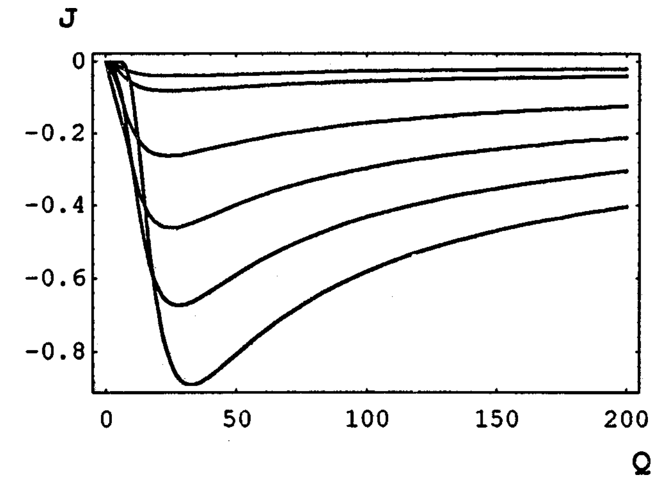

FIG. 3. Steady current J vs the multiplicative noise strength Q. From top to bottom, the curves corresponding to the value of noise correlation strength λ: -0.05, -0.1, -0.3, -0.5, -0.7, -0.9, respectively; $L=1$, $C=\frac{1}{4}$, $d=\frac{1}{2}$, $D=0.3$.

$$A_6=\frac{L^2}{4d}\left[e^{-2d/k_2}+\frac{k_2}{d}(e^{d/k_2}-1)\right].$$

In order to illustrate the characteristics of fluctuation-induced current, we plot in Figs. 1–3 the dependence of J in Eq. (15) upon noise parameters. The important points are observed from the figures: (1) Current reversal. $J\sim\lambda$ curves in Fig. 1 pass through the origin and thus the direction of current J reverses when the sign of the multiplicative and additive noise correlation λ is changed. (2) Existence of extremum. The direction of current J is positive when $\xi(t)$ and $\eta(t)$ are positively correlated, i.e., $\lambda>0$. In this case, the dependence of current upon the multiplicative noise intensity Q is nonlinear and possesses a maximum (see Fig. 2). For the case of negative noise correlation $\lambda<0$, the current J is negative and possesses a minimum in its dependence on Q (see Fig. 3). In the model system discussed above, transport current exists even though the Brownian particle is in a periodic potential that is symmetrical in space. The current appears to be induced by the correlation between the two Gaussian white noises. Why does the noise correlation induce transport? The origin lies in the generalized potential $\Phi(x)$ that is tilted. It is easy to verify that $\Phi(x+L)-\Phi(x)=-8\lambda\sqrt{QD}Cd/k_1k_2$. Obviously, $\Phi(x)$ is tilted as long as λ is not equal to 0. The current J arises consequently. Another point is that the tilt $\Phi(x+L)-\Phi(x)$ changes its sign when λ does so, which is the origin of the current reversal Furthermore, the mechanism of the fluctuation-induced transport can be physically depicted as follows. The multiplicative noise (MN) makes the potential fluctuate

$$U_F(x)=\begin{cases}[2d/L-C\xi(t)]x-2(n-1)d & \text{when } (n-1)L\leqslant x\leqslant(2n-1)L/2\\ -[2d/L-C\xi(t)]x+2nd & \text{when } (2n-1)L/2\leqslant x\leqslant nL.\end{cases}$$

When the MN assumes a positive realization, the slope of the fluctuating potential $U_F(x)$ is reduced. Inversely, when MN assumes a negative realization, the slope of $U_F(x)$ is enhanced. This fact makes the mean current zero. However, the MN $\xi(t)$ is correlated to the additive noise $\eta(t)$. A positive correlation $(\lambda>0)$ implies that probability for $\xi(t)$ to assume a positive value is greater when $\eta(t)>0$ and that probability for $\xi(t)$ to assume a negative value is greater when $\eta(t)<0$. [16] Since both $\xi(t)$ and $\eta(t)$ are white noises, they fluctuate on the same time scale and thus enhance each other statistically. Therefore, driven by the additive noise and the fluctuating potential $U_F(x)$ (caused by MN), the motion of the Brownian particle points toward the positive direction on the average. On the other hand, a negative correlation $\lambda<0$ implies that it is more probable for $\xi(t)$ to assume a negative value when $\eta(t)>0$ and that it is more probable for $\xi(t)$ to assume a positive value when $\eta(t)<0$. An analysis similar to the case of $\lambda>0$ shows that the motion of the Brownian particle points toward the negative direction on the average. Here a conclusion can be easily drawn: The correlation between the multiplicative noise and the additive noise not only induces directed motion (nonzero average current) but also determines its direction.

In summary, we have presented an analytical solution for the stochastic motion of a Brownian particle in a periodic potential subject to a multiplicative noise and an additive noise. Detailed analysis of a model system shows the correlation between the two noises can induce a nonzero current.

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